Proc. Indian Acad. Sci. (Math. Sci.) Vol. 116, No. 3, August 2003, pp. 317–323.
Printed in India

Joint local quasinilpotence and common invariant subspaces

A FERNÁNDEZ VALLES

Department of Mathematics, University of Cádiz, Avda. de la Universidad,
s/n 11402-Jerez de la Frontera, Spain
E-mail: aurora.fernandez@uca.es

MS received 27 November 2005; revised 3 February 2006

Abstract. In this article we obtain some positive results about the existence of a common nontrivial invariant subspace for N-tuples of not necessarily commuting operators on Banach spaces with a Schauder basis. The concept of joint quasinilpotence plays a basic role. Our results complement recent work by Kosiek [6] and Ptak [8].

Keywords. Joint local quasinilpotence; uniform joint local quasinilpotence; common invariant subspaces; positive operators.

1. Introduction

Let T be a continuous linear operator defined on a separable Banach space X. Let us say that T is cyclic if $x \in X$ such that

$$\text{Linear Span}\{T^n x \colon x \in X\}$$

is dense in X.

On the other hand, we said that T is locally quasinilpotent at $x \in X$ if

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = 0.$$

The notion of local quasinilpotence was introduced in [1] to obtain nontrivial invariant subspaces for positive operators.

Positive results about N-tuples of operators with a nontrivial common invariant subspace have been obtained in [2,4,7,8]. In this article, we extend the results of Abramovich *et al* [1] to the case of not-necessarily commuting N-tuples of continuous linear operators.

To extend these results it will be essential to introduce the notion of joint local quasinilpotence for N-tuples of operators (not necessarily commuting). The results complement the results obtained in [1].

The main result of this paper appears in §3 and is stated as follows:

Theorem 3.1. *Let $T = (T_1, \ldots, T_N)$ be a N-tuple of continuous linear operators. If T is positive with respect to a cone C and $x_0 \in C$ exists such that T is joint locally quasinilpotent at x_0, then there exists a common nontrivial invariant subspace for $T = (T_1, \ldots, T_N)$.*

Moreover, using this theorem we deduce new results on nontrivial common invariant subspaces for N-tuples of operators (see Theorem 3.2, Corollary 3.3). We will conclude this article with a section including open problems and further directions.

2. Joint local quasinilpotence

It is easy to see that if $(T_1,\ldots,T_N)$ are N commuting operators and they are locally quasinilpotent at $x_0 \in X$, then the compositions $T_{i_1}\ldots T_{i_m}; 1 \leq i_j \leq N$ for all $j \in \{1,\ldots,m\}$ and all $m \in \mathbb{N}$, are locally quasinilpotent at x_0. In fact the intersection of the sets

$$Q_{T_i} = \{x \in X, \text{ such that } T_i \text{ is locally quasinilpotent at } x\},$$

is a common invariant manifold.

However if $T_1,\ldots,T_N$ are not commuting, the problem becomes more complicated.

Example. Let T_1, T_2 be two operators on the Hilbert space l_2 defined in the following way:

$$T_1 e_n = \begin{cases} e_{n-1}, & \text{if } n \geq 2 \\ 0, & \text{if } n = 1 \end{cases} ; \quad T_2 e_n = \frac{1}{n} e_{n+1},$$

where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of l_2.

The operators T_1 and T_2 are locally quasinilpotent at e_k for each $k \geq 2$, since $T_1^n e_k = 0$ for each $n \geq k$, and therefore $\lim_{n\to\infty} \|T_1^n e_k\|^{\frac{1}{n}} = 0$. On the other hand, $T_2^n e_k = \frac{1}{k(k+1)\cdots(k+n-1)} e_{n+k}$, hence

$$\lim_{n\to\infty} \|T_2^n e_k\|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{k(k+1)\cdots(k+n-1)}\right)^{1/n} = 0$$

and therefore T_1 and T_2 are locally quasinilpotent at e_k for each $k \geq 2$.

However, T_1T_2 and T_2T_1 are not locally quasinilpotent at e_k for each $k \geq 2$. Indeed, since $T_1T_2e_k = \frac{1}{k}e_k$,

$$\lim_{n\to\infty} \|(T_1T_2)^n e_k\|^{1/n} = \lim_{n\to\infty} \left\| \left(\frac{1}{k}\right)^n e_k \right\|^{1/n} = \frac{1}{k} \neq 0.$$

On the other hand, we know $T_2T_1e_k = \frac{1}{k-1}e_k$ and hence $\lim_{n\to\infty} \|(T_2T_1)^n e_k\|^{1/n} = \frac{1}{k-1} \neq 0$.

Let $T = (T_1,\ldots,T_N)$ be an N-tuple of continuous linear operators on a Banach space X not necessarily commuting. Let us denote by T^n the collection of all possible products of n elements in T.

DEFINITION 2.1.

Let $T = (T_1,\ldots,T_N)$ be an N-tuple of continuous linear operators on a Banach space X not necessarily commuting. Then, we will say that T is uniform joint locally quasinilpotent at $x_0 \in X$ if

$$\lim_{n\to\infty} \max_{S\in T^n} \|Sx_0\|^{1/n} = 0.$$

The notion of uniform joint local quasinilpotence is closely related with the joint spectral radius defined by Rota and Strang [9]. We can get more information about spectral theory in several variables in [7].

Although the results of this article are formulated under the hypothesis of uniform joint local quasinilpotence, nevertheless, sometimes it will be possible to replace the above-mentioned hypothesis by the following weaker property.

DEFINITION 2.2.

Let $T = (T_1, \ldots, T_N)$ be an N-tuple of continuous linear operators on a Banach space X. Then we will say that T is joint locally quasinilpotent at $x_0 \in X$ if

$$\lim_{n\to\infty} \|T_{i_1}T_{i_2}\cdots T_{i_n}x_0\|^{1/n} = 0,$$

for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$.

The difference between the concept of uniform joint local quasinilpotence and joint local quasinilpotence is the lack of uniformity. Next we see some properties of both concepts.

PROPOSITION 2.3.

Let $T = (T_1, \ldots, T_N)$ be an N-tuple of continuous linear operators on a Banach space X and let us suppose that T is uniform joint locally quasinilpotent at $x_0 \in X\setminus\{0\}$. Then for all polynomial p of m variables, such that $p(0, \ldots, 0) = 0$ we have that

$$\lim_{n\to\infty} \|p(T_{i_1}, \ldots, T_{i_m})^n x_0\|^{1/n} = 0,$$

where $i_j \in \{1, \ldots, N\}$; $j \in \{1, \ldots, m\}$, that is, the operator $p(T_{i_1}, \ldots, T_{i_m})$ is locally quasinilpotent at x_0.

Proof. Fix $\varepsilon > 0$. Let us suppose that $k \in \mathbb{N}$ is the number of summands of the polynomial p, and let us denote by $c \in \mathbb{R}_+$ the maximum of the modulus of the coefficients of p. Then, since $T = (T_1, \ldots, T_N)$ is uniform joint locally quasinilpotent at x_0, there exists $n_0 \in \mathbb{N}$ such that

$$\max_{S\in T^n} \|Sx_0\|^{1/n} < \frac{\varepsilon}{ck}$$

for all $n \geq n_0$.

Now, taking into account that the polynomial p has no independent term, for all $n \geq n_0$,

$$\|p(T_1, \ldots, T_N)^n x_0\|^{1/n} \leq (k^n c^n \max_{S\in T^n} \|Sx_0\|)^{1/n} \leq \varepsilon,$$

which proves the desired result. □

Remark 2.4. In fact the condition on the polynomial $p(\theta) = 0$ is a necessary condition in the proof of Proposition 2.3. Indeed, let F be the forward shift defined on the sequences space ℓ_2 by $Fe_n = \frac{1}{n!}e_{n+1}$, for all $n \geq 1$. It is easy to see that the operator $I + F$ is not locally quasinilpotent at e_1 (where I denotes the identity operator).

Definitions 2.1 and 2.2 are the natural extensions of the notion of local quasinilpotence for N-tuples of continuous linear operators. In fact, let us denote

$$Q_{T_1\ldots T_N} = \{x_0 \in X\colon (T_1 \ldots T_N) \text{ is joint locally quasinilpotent at } x_0\}$$

and let us denote by $UQ_{T_1\ldots T_N}$ the set of all uniform joint locally quasinilpotent vectors for $(T_1, \ldots, T_N)$. Then, we have the following result.

PROPOSITION 2.5.

Let $T=(T_1,\dots,T_N)$ be an N-tuple of continuous linear operators on a Banach space X, then the sets $UQ_{T_1\dots T_N}$ and $Q_{T_1\dots T_N}$ are common invariant manifolds by $T_1,\dots,T_N$.

Proof. Clearly, $x\in Q_{T_1\dots T_N}$ implies that $\lambda x\in Q_{T_1\dots T_N}$. Now let $x,y\in Q_{T_1\dots T_N}$, and fix $\varepsilon>0$. Then, there exists some n_0 such that $\|T_{i_1}\dots T_{i_n}x\|<\varepsilon^n$ and $\|T_{i_1}\dots T_{i_n}y\|<\varepsilon^n$ for each $i_1,\dots,i_n\in\{1,\dots,N\}$ and each $n\ge n_0$. Therefore,

$$\|T_{i_1}\dots T_{i_n}(x+y)\|^{1/n}\le(\|T_{i_1}T_{i_2}\dots T_{i_n}x\|+\|T_{i_1}T_{i_2}\dots T_{i_n}y\|)^{1/n}<2\varepsilon$$

for all $n\ge 0$. Therefore $x+y\in Q_{T_1\dots T_N}$ and so $Q_{T_1\dots T_N}$ is a vector manifold.

Fix $x_0\in Q_{T_1\dots T_N}$ and let $T_kx_0\in Q_{T_1,\dots,T_N}$. Then

$$\lim_{n\to\infty}\|T_{i_1}\dots T_{i_n}(T_kx_0)\|^{1/n}=\lim_{n\to\infty}(\|T_{i_1}\dots T_{i_m}T_kx_0\|^{\frac{1}{n+1}})^{\frac{n+1}{n}}=0$$

for each $i_j\in\{1,\dots,N\}$; $j\in\mathbb{N}$ and for each $k\in\{1,\dots,N\}$. Therefore $Q_{T_1\dots T_N}$ is a common invariant manifold for $T_1,\dots,T_N$. Similar proof follows for the set $UQ_{T_1\dots T_N}$, and hence we omit it. $\square$

The above propositions show that if $Q_{T_1\dots T_N}\ne\{0\}$ ($UQ_{T_1,\dots,T_N}\ne\{0\}$ respectively) and $\overline{Q_{T_1\dots T_N}}\ne X$ ($\overline{UQ_{T_1\dots T_N}}\ne X$ respectively), then $\overline{Q_{T_1\dots T_N}}$ ($\overline{UQ_{T_1\dots T_N}}$ respectively) is a common nontrivial closed invariant subspace for $T_1,\dots,T_N$. As far as the invariant subspace problem is concerned, we need only consider the two extreme cases $\overline{Q_{T_1\dots T_N}}=X$ and $Q_{T_1\dots T_N}=\{0\}$.

3. Main results

Let X be a Banach space with a Schauder basis (x_n). We say that $T=(T_1,\dots,T_N)$ is positive with respect to the cone

$$C=\left\{\sum_{j=1}^{\infty}t_jx_j:\quad t_j\ge 0\right\}$$

if $T_j(C)\subset C$ for all $j\in\{1,\dots,N\}$.

Let us see the main result of this paper.

Theorem 3.1. *Let $T=(T_1,\dots,T_N)$ be an N-tuple of continuous linear operators on a Banach space with a Schauder basis such that $T=(T_1,\dots,T_N)$ is positive with respect to the cone C, and let us suppose that $y_0\in C$ exists such that $T=(T_1,\dots,T_N)$ is joint locally quasinilpotent at y_0. Then there exists a common nontrivial invariant subspace for $T=(T_1,\dots,T_N)$.*

Proof. Let (x_n) be a Schauder basis of the Banach space X and let (f_n) be the sequence of coefficient functionals associated with the basis (x_n).

Assume that $T=(T_1,\dots,T_N)$ is joint locally quasinilpotent at some $y_0\in C\setminus\{0\}$, i.e., $\lim_{n\to\infty}\|T_{i_1}\dots T_{i_n}y_0\|^{1/n}=0$ with $i_j\in\{1,\dots,N\}$ for all $j\in\mathbb{N}$. Let us suppose that $T_iy_0=0$ for all $i\in\{1,\dots,N\}$. Then $\bigcap_{i=1}^{N}\ker(T_i)$ is a common nontrivial invariant subspace for each $T_1,\dots,T_N$. Thus, we can suppose that $T_{i_0}y_0\ne 0$ for some $i_0\in\{1,\dots,N\}$. By an

appropriate scaling of y_0, we can assume that $0 < x_k \le y_0$ and $T_{i_0}x_k \neq 0$ for some k and some $i_0 \in \{1,\dots,N\}$.

Now let $P\colon X \to X$ denote the continuous projection onto the vector subspace generated by x_k defined by $P(x) = f_k(x)x_k$. Clearly, $0 \le P(x) \le x$ holds for each $0 < x \in X$. We claim that

$$PT_{i_1}\dots T_{i_m}x_k = 0 \tag{1}$$

for each $m \ge 0$. To see this, fix $m \ge 0$ and let $PT_{i_1}\dots T_{i_m}x_k = \alpha x_k$ for some nonnegative scalar $\alpha > 0$. Since P is a positive operator and the composition of positive operators is a positive operator, it follows that

$$0 < \alpha^n x_k = (PT_{i_1}\dots T_{i_m})^n x_k \le (T_{i_1}\dots T_{i_m})^n x_k \le (T_{i_1}\dots T_{i_m})^n y_0.$$

Let us observe that the following inequality is not true because the norm is not monotone $\alpha^n\|x_k\| \le \|(T_{i_1}\dots T_{i_m})^n y_0\|$. However, if we use the fact that f_k is a positive linear functional, we conclude that

$$0 < \alpha^n = f_k(\alpha^n x_k) \le f_k((T_{i_1}\dots T_{i_m})^n y_0).$$

Consequently, $0 < \alpha^n \le \|f_k\|\|(T_{i_1}\dots T_{i_m})^n y_0\|$, and so

$$0 < \alpha \le \|f_k\|^{1/n}\|(T_{i_1}\dots T_{i_m})^n y_0\|^{1/n}.$$

From Definition 2.2 we know $\lim_{n\to\infty}\|(T_{i_1}\dots T_{i_m})^n y_0\|^{1/n} = 0$. Thus we deduce that $\alpha = 0$, and condition (1) must be true.

Now let us consider the linear subspace Y of X generated by the set

$$\{T_{i_1}\dots T_{i_m}x_k\colon m \in \mathbb{N}; i_j \in \{1,\dots,N\} \quad \text{for all } j \in \mathbb{N}\}.$$

Clearly, Y is invariant for each $T_j; j \in \{1,\dots,N\}$ and, since $0 \neq T_{i_0}x_k \in Y$ for some $i_0 \in \{1,\dots,N\}$, we see that $Y \neq \{0\}$. From (1) it follows that $f_k(T_{i_1}\dots T_{i_m}x_k)x_k = P(T_{i_1}\dots T_{i_m}x_k) = 0$, therefore $f_k(T_{i_1}\dots T_{i_m}x_k)$ for each $i_1,\dots,i_{\underline{m}}$. This implies that $f_k(y) = 0$ for each $y \in Y$, and consequently $f_k(y) = 0$ for all $y \in \overline{Y}$, that is, $\overline{Y} \neq X$. The latter shows that $\overline{Y}$ is a common nontrivial closed invariant subspace for the operators $T_j; j \in \{1,\dots,N\}$, and the proof is complete. □

Let $T_1,\dots,T_N$ be joint locally quasinilpotent operators at $x_0 \in C$. Then we can add arbitrary weights to each matrix representing the operators $T_1,\dots,T_N$ and still guarantee the existence of a common nontrivial closed invariant subspace.

Remark 3.2.

(a) First, let us observe that if $(T_1,\dots,T_N)$ is joint locally quasinilpotent at x_0 it is possible to obtain a closed invariant subspace F (nontrivial) invariant also for every positive operator A such that $AT_i = T_iA$ for all $i \in \{1,\dots,N\}$. Indeed, the above proof follows considering the closed subspace generated by

$$\begin{aligned}&\{AT_{i_1}\dots T_{i_m}x_k\colon m \in \mathbb{N}; i_j \in \{1,\dots,N\}\\ &\quad \forall\ j \in \mathbb{N}, A \text{ positive } AT_i = T_iA(\forall i)\}.\end{aligned}$$

(b) On the other hand, let us mention that the subspace guaranteed in the above theorem is in fact an invariant nontrivial ideal.

In the following theorem, positivity is with respect to the cone generated by the Schauder basis of the Banach space.

Theorem 3.3. *Let X be a Banach space with a Schauder basis. Assume that the matrix $A_k = (a^k_{ij})$ defines a continuous operator T_k for all $k \in \{1,\dots,N\}$, such that the N-tuple $T = (T_1,\dots,T_N)$ is joint locally quasinilpotent at a nonzero positive vector. Let $(w^k_{ij}); k \in \{1,\dots,N\}$ be N-double sequences of complex numbers. If the weighted matrices $B_k = (w^k_{ij}a^k_{ij}); k \in \{1,\dots,N\}$ define continuous operators $B_k; k \in \{1,\dots,N\}$, then $B_1,\dots,B_N$ have a common nontrivial closed invariant subspace.*

Proof. Let (x_n) be a Schauder basis of the Banach space X, and let (f_n) be the sequence of functional coefficients associated with the basis (x_n). Assume that the operators $T_1,\dots,T_N$ satisfy $\lim_{n\to\infty}\|T_{i_1}\dots T_{i_n}y_0\|^{1/n} = 0; i_j \in \{1,\dots,N\}; j \in \mathbb{N}$ for some positive nonzero vector y_0. An appropriate scaling of y_0 shows that there exists l satisfying $0 < x_l \le y_0$. Let us suppose that $T_k x_l = 0$ for all $k \in \{1,\dots,N\}$, then an easy argument shows that $B_k x_l = 0$ for all $k \in \{1,\dots,N\}$, and $\bigcap_{k=1}^N \ker(B_k)$ is a nontrivial closed invariant subspace (here we assume that $B_k \ne 0$ for all $k \in \{1,\dots,N\}$). Thus, we can suppose that $T_{i_0}x_l \ne 0$ for some $i_0 \in \{1,\dots,N\}$.

Now, let us denote by $P\colon X \to X$, the positive projection defined by $P(x) = f_l(x)x_l$. Arguing as in the proof of Theorem 3.1, we can establish that $PT_{i_1}\dots T_{i_m}x_l = 0$ for each $m \in \mathbb{N}$, where $i_j \in \{1,\dots,N\}$ for all $j \in \mathbb{N}$. In particular, we have $f_l(T_{i_1}\dots T_{i_m}x_l) = 0$ for each $i_1,\dots,i_m$. Consequently, for each $m \in \mathbb{N}$ and for each positive operator $S\colon X \to X$ satisfying $0 \le S \le T_{i_1}\dots T_{i_m}$, we have

$$0 \le f_l(Sx_l) \le f_l(T_{i_1}\dots T_{i_m}x_l) = 0. \tag{2}$$

Next, let us consider the vector subspace Y generated by the set

$$\{Sx_l\text{: such that } S \text{ satisfies } 0 \le S \le T_{i_1}\dots T_{i_m} \text{ for some } i_1,\dots,i_m\}.$$

Clearly Y is invariant for each operator $R_k; k \in \{1,\dots,N\}$ satisfying $0 \le R_k \le T_k$. Also, from (2) it follows that

$$f_l(y) = 0$$

for all $y \in \overline{Y}$, that is, $\overline{Y} \ne X$. The latter shows that $\overline{Y}$ is a nontrivial closed vector subspace of X. Let $A^k_{ij}; k \in \{1,\dots,N\}$ now be the operators defined by

$$A^k_{ij}(x_j) = a^k_{ij}x_j \quad \text{and} \quad A^k_{ij}(x_m) = 0 \quad \text{for } m \ne j.$$

Since A^k_{ij} satisfy $0 \le A^k_{ij} \le A_k$ for all $k \in \{1,\dots,N\}$, it follows that $\overline{Y}$ is invariant for each one of the operators A^k_{ij}. Therefore, the vector subspace $\overline{Y}$ is invariant under the operators

$$B^k_n = \sum_{i=1}^n \sum_{j=1}^n w^k_{ij}A^k_{ij}.$$

However, the sequence of operators $(B^k_n); k \in \{1,\dots,N\}$ converges in the strong operator topology to B_k. Therefore, $B_k(\overline{Y}) \subset \overline{Y}$ and, thus, the operators $B_1,\dots,B_N$, have a common nontrivial closed invariant subspace. □

COROLLARY 3.4.

Let X be a Banach space with a Schauder basis. Assume that the positive matrices $A_k = (a_{ij}^k)$ define continuous operators on X, which are joint locally quasinilpotent at a nonzero positive vector. If the continuous operators $T_k\colon X \to X$ are defined by the matrices $T_k = (t_{ij}^k)$ satisfying $t_{ij}^k = 0$ whenever $a_{ij}^k = 0$, then the operators have a common nontrivial closed invariant subspace.

4. Concluding remarks and open problems

The notion of uniform joint local quasinilpotence is used extensively in [5] to obtain common nontrivial invariant subspaces. Both concepts, joint local quasinilpotence and uniform joint local quasinilpotence, play an important role in the search of common nontrivial invariant subspaces.

It would be interesting to know something more on the sets $\overline{Q_{(T_1,\dots,T_N)}}$ and $\overline{UQ_{(T_1,\dots,T_N)}}$. Our conjecture is that both sets are equal in majority of the cases.

On the other hand, it would be interesting to extend the results of Theorems 3.1 and 3.3 for the case of N-tuples of positive operators defined on a Hausdorff topological vector space, where the partial order is defined by means of a Markushevish basis.

Acknowledgement

The authors would like to thank the referee for his comments and suggestions that have improved this article.

References

[1] Abramovich Y A, Aliprantis C D and Burkinshaw O, Invariant subspace theorems for positive operators, *J. Funct. Anal.* **124(1)** (1994) 95–111

[2] Bercovici H, Commuting power-bounded operators, *Acta Sci. Math. (Szeged)* **57(1–4)** (1993) 55–64

[3] Fernández Valles A, Common non-supercyclic vectors for commuting contractions, Preprint UCA (2006) pp. 1–7

[4] Fernández Valles A, Common nontrivial closed cones for commuting contractions, Preprint UCA (2006) pp. 1–8

[5] Fernández Valles A, Invariant subspaces for uniform joint locally quasinilpotent operators, Preprint UCA (2006) pp. 1–12

[6] Kosiek M and Octavio A, On common invariant subspaces for commuting contractions with rich spectrum, *Indiana Univ. Math. J.* **53(3)** (2004) 823–844

[7] Müller V, Spectral theory of linear operators and spectral systems in Banach algebras, in: Operator Theory: Advances and Applications (Basel: Birkhäuser Verlag) (2003) vol. 139

[8] Ptak M, On the existence of invariant subspaces and reflexivity of N-tuples of operators, *Taiwanese J. Math.* **1(3)** (1997) 231–290

[9] Rota G C and Strang G, A note on the joint spectral radius, *Indag. Math.* **22** (1960) 379–381